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DEDICATION

This work is dedicated

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Bouras Hedda

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Introduction

It is diffcult to localize the first moment when asymmetric norms were used, but it goes back as early as 1968 in a paper by Duffin and Karlovitz. Also in 1973 Krein and Nudelman used asymmetric norms in their study of some problems. In 2005 García-Raffi began the study about compactness in this space.

The main goal of this memoir is to study the compactness on asymmetric normed linear spaces. The aim of the Chapter 1 is to present the basic results on asymmetric normed spaces.

After this prelimnaries, we will present in Chapter 2, general results regarding compactness in asymmetric normed spaces. In this chapter we have two section, the aim of first section is to extend the results about compact sets on finite dimensional normed spaces to the case of asymmetric normed linear spaces, where we give the set theoretical arguments that allows to a general description of compact sets of an asymmetric normed linear space. The second section is devoted to characterizing the compactness and precompactness of subsets on asymmetric normed linear spaces.

The aim of the last chapter is to study some properties of continuous and compact operators on asymmetric normed spaces and to study some of their properties. This chapter contains also a short discussion on dual space of an asymmetric normed linear space.

Notations

- $\mathbb{K}$: The field of real or complex numbers;
- p : Be an asymmetric norm on X over $\mathbb{R}^+$;
- $d_p(x, y) = p(y - x)$: The quasi-metric associated to an asymmetric norm p;
- p^s : Be a norm on X over $\mathbb{R}^+$;
- $B_r^p(x_0) = \{x \in X : p(x - x_0) < r\}$: An open ball in an asymmetric normed space(X, p);
- $\overline{B}_r^p(x_0) = \{x \in X : p(x - x_0) \leq r\}$: A closed ball in an asymmetric normed space (X, p);
- $B_p = \{x \in X : p(x) < 1\}$: The open unit ball of an asymmetric normed space (X, p);
- $\overline{B}_p = \{x \in X : p(x) \leq 1\}$: The closed unit ball of an asymmetric normed space (X, p);
- $B_{p^s} = \{x \in X : p^s(x) < 1\}$: The open unit ball of a normed space (X, p^s);
- $\overline{B}_{p^s} = \{x \in X : p^s(x) \leq 1\}$: The closed unit ball of a normed space (X, p^s);
- $\mathcal{LC}(X, Y)$: The set of all continuous linear maps from (X, p) to (Y, q);
- $\mathcal{LC}^s(X, Y)$: The set of all continuous linear maps from (X, p^s) to (Y, q^s);
- $\mathcal{LC}^k(X, Y)$: The set of all linear compact operators from (X, p) to (Y, q);
- X^* : The dual space of an asymmetric normed space (X, p);
- X^{*s} : The dual space of a normed space (X, p^s).

Chapter 1

Asymmetric normed space

The goal of this chapter is to present the definition of asymmetric normed spaces. Since the basic topological tools come from quasi-metric spaces, this chapter contains a short presentation of some fundamental results of this space. The focus is on those which are most used in functional analysis - Balls, Bases of neighbourhoods and Convergence (ect). For a good presentation of the compactness on asymmetric normed spaces we have mentioned the compactness in the normed spaces.

Our basic reference for asymmetric normed linear spaces is [3].

1.1 Prelimnaries on compactness in the normed space

Definition 1.1.1 *Let X be a normed space. The space X is **compact** provided that every open cover of X has a finite subcover, that is for every collection O of open sets whose union equals X, there is a finite subcollection $\{O_i\}_{i=1}^{n}$ of O whose union equals X.*

Theorem 1.1.1 *The normed space X is compact if and only if every sequence in X has a convergent subsequence.*

Theorem 1.1.2 *[13, Vol. I, Proposition 1.e.2] A subset K of X is relatively compact if and only if there is a sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $\|x_n\| \longrightarrow 0$ and $K \subset \overline{convex}\,\{x_n\}_{n=1}^{\infty}$.*

Theorem 1.1.3 *The normed space X is finite dimensional if and only if its closed unit ball is compact.*

Definition 1.1.2 *Let X be a normed space. We say that a subset K of X is **precompact** if for all $\varepsilon > 0$, K can be covered by a finite number of sets of diameter less than or equal to ε.*

Theorem 1.1.4 *Let X be a normed space. The following statements hold:*

1. *If X is compact, then X is precompact.*
2. *X is compact if and only if X is precompact and complete.*

1.2 Asymmetric normed spaces

Definition 1.2.1 *Let X be a vector space over the fieled $\mathbb{K}$. An asymmetric norm on X is a function $p : X \longrightarrow \mathbb{R}^+$ such that*

1. $p(x) = p(-x) = 0$ if and only if $x = 0$ for all $x \in X$.
2. $p(\lambda x) = \lambda p(x)$ for all $x \in X$ and $\lambda \in \mathbb{R}^+$.

3. Triangle inquality: $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$.

We will also consider extended asymmetric norms, i.e. taking values in $[0, +\infty]$.

A vector space X together with an asymmetric norm p is called an asymmetric normed linear space. The notion of an extended asymmetric normed linear space is defined in the obvious manner.

If, further,

$$p(x) = p(-x) = 0 \nRightarrow x = 0,$$

a map p is called an asymmetric seminorm.

Here, we call the pair (X, p), an asymmetric seminormed linear space.

Proposition 1.2.1 *Let (X, p) be an asymmetric normed linear space.*

- The function $\overline{p} : X \longrightarrow \mathbb{R}^+$ defined on X by

$$\overline{p}(x) := p(-x) \text{ for all } \quad x \in X,$$

  is an asymmetric norm on X, called the conjugate of p.

- The function $p^s : X \longrightarrow \mathbb{R}^+$ defined on X by

$$p^s(x) := \max\{p(x), p(-x)\} \text{ for all } \quad x \in X,$$

  is a norm on X, called the associated norm of p.

- For all $x, y \in X$, the following inequalities hold

$$|p(x) - p(y)| \leq p^s(x-y) \text{ and } |\overline{p}(x) - \overline{p}(y)| \leq p^s(x-y).$$

Note that, An asymmetric norm on X is a norm; but the converse is not true in general. In order to prove this we take the following example.

Example 1.2.1 *The function $p : \mathbb{R} \longrightarrow \mathbb{R}^+$ given by the formula*

$$p(\alpha) = \max(\alpha, 0),$$

is an asymmetric norm. In addition,

$$\overline{p}(\alpha) = p(-\alpha) = \max\{-\alpha, 0\},$$

and

$$p^s(\alpha) = \max\{p(\alpha), p(-\alpha)\} = |\alpha|.$$

1.3 Quasi-metric spaces

Let X be a vector space over the fieled $\mathbb{K}$.

Definition 1.3.1 *A quasi-metric is a function $d : X \times X \longrightarrow \mathbb{R}^+$ with the following properties*

1. $d(x, y) = d(y, x) = 0$ if and only if $x = y$ for all $x, y \in X$,
2. Triangle inquality: $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

The pair (X, d) is called quasi-metric space.

If

$$d(x, y) = d(y, x) = 0 \nRightarrow x = y,$$

for some $x, y \in X$, the function d is called a quasi-semimetric, and the pair (X, d) is called a quasi-semimetric space.

Remark 1.3.1 *If d is a quasi-metric on $X \times X$, then the function $\overline{d}$ defined on $X \times X$ by*

$$\overline{d}(x, y) = d(y, x) \qquad \textit{for all } x, y \in X,$$

is a quasi-metric on X called the conjugate of d.

And the function d^s defined on $X \times X$ by

$$d^s(x, y) = \max\{d(x, y), \overline{d}(x, y)\} \qquad \text{for all } x, y \in X,$$

is a metric on X.

Let (X, d) be a quasi-metric space.

Definition 1.3.2 (Balls) *Let $x_0 \in X$ and $r > 0$. The open ball, of radius r centered at x_0, to be the set*

$$B_{X,d}(x_0, r) = \{x \in X : d(x_0, x) < r\},$$

and define the closed ball of radius r centered at x_0 to be the set

$$B_{X,d}[x_0, r] = \{x \in X : d(x_0, x) \leq r\}.$$

Let $x \in X$, we say that a set $V \subset X$ is a neighborhood of the point x if and only if

$$\exists r > 0 : B_{X,d}(x, r) \subset V.$$

The collection neighborhoods of the point x is denoted by $V_p(x)$.

The topology τ_d of a quasi-metric space (X, d) can be defined starting from the family $V_p(x)$.

Definition 1.3.3 *The convergence of a sequence $(x_n)_n$ to x with respect to τ_d, called d-convergence, in symbols $x_n \xrightarrow{d} x$, can be characterized in the following way:*

$$x_n \xrightarrow{d} x \Longleftrightarrow d(x, x_n) \longrightarrow 0.$$

Proposition 1.3.1 *[3, Page 4] Let $(x_n)_n$ be a sequence in a quasi-metric space (X, d).*

1. If $(x_n)_n$ is d-convergent to x and $\overline{d}$-convergent to y, then $d(x, y) = 0$.

2. If $(x_n)_n$ is d-convergent to x and $d(y, x) = 0$, then $(x_n)_n$ is also d-convergent to y.

Remark 1.3.2 *Every asymmetric norm p, on a linear space X, induces a quasi-metric d_p on $X \times X$ defined by*

$$d_p(x, y) = p(y - x), \quad x, y \in X.$$

If p is an asymmetric norm on X, then the topology τ_{d_p} will be simply denoted by τ_p and we will say that τ_p is the topology induced by p.

1.4 Some topological properties of an asymmetric normed space

Let X be a vector space over the fieled $\mathbb{K}$.

First we point out that, most of what was said in this part was quoted from the book [3].

Definition 1.4.1 *Let $x_0 \in X$ and $r > 0$. The subsets*

$$B_r^p(x_0) = \{x \in X : p(x - x_0) < r\},$$

and

$$\overline{B}_r^p(x_0) = \{x \in X : p(x - x_0) \leq r\},$$

are respectively called the open and closed balles centred at x with radius r.

Remark 1.4.1 *Denoted by $B_p = B_1^p(0), \overline{B}_p = \overline{B}_1^p(0)$ the unit balls. In this case the following equalities hold*

$$B_r^p(x_0) = x_0 + rB_p \qquad \text{and} \qquad \overline{B}_r^p(x_0) = x_0 + r\overline{B}_p.$$

Definition 1.4.2 *Let U be a subset of X. We say U is τ_p-**open set** if for each $x \in U$ there is an open ball centered at x and contained in U.*

Remark 1.4.2 *The topology τ_{p^s} is finer than the topologies τ_p and $\tau_{\overline{p}}$. This means that, any τ_p-open (closed) set is τ_{p^s}-open (closed); similar results hold for the topology $\tau_{\overline{p}}$.*

Proposition 1.4.1 *If (X, p) is an asymmetric space, then any ball $B_r^p(x_0)$ is τ_p-open and any ball $\overline{B}_r^p(x_0)$ is $\tau_{\overline{p}}$-closed.*

Also, the following inclusions hold:

$$B_r^{p^s}(x_0) \subset B_r^p(x_0) \text{ and } B_r^{p^s}(x_0) \subset B_r^{\overline{p}}(x_0),$$

with similar inclusions for the closed balls.

Notation: In the sequel we will get the following notations:

- $V_\varepsilon(x) = B_\varepsilon^p(x) = \{y \in X : p(y - x) < \varepsilon\}$.
- $\theta_x = \{y \in X, d_p(x, y) = p(y - x) = 0\}$.

Let (X, p) be an asymmetric norm space.

Definition 1.4.3 *We say that a set $V \subset X$ is a **neighborhood** of the point $x \in X$ if*

$$\exists r > 0 : B_r^p(x) \subset V.$$

Proposition 1.4.2 *The sets*

$$V_\varepsilon(0) := \{x \in X : p(x) < \varepsilon\},\ \varepsilon > 0$$

form a base of neighbourhoods of zero (or fundamental system of neighborhood of zero) for the topology τ_p generated by d_p. Hence the sets

$$V_\varepsilon(x) = x + V_\varepsilon(0),$$

from a fundamental system of neighborhood of x for all $x \in X$.

Definition 1.4.4 (Banach lattice space) *[13] A partially ordered Banach space X over the reals is called a Banach lattice provided*

1. *$x \leq y$ implies $x + z \leq y + z$, for every $x, y, z \in X$.*
2. *$\lambda x \geq 0$, for every $x \geq 0$ in X and every non negative real λ.*
3. *for all $x, y \in X$ there exists a least upper bound (l.u.b.) $x \vee y = \sup\{x, y\}$ and a greatest lower bound (g.l.b.) $x \wedge y = \inf\{x, y\}$.*
4. *$\|x\| \leq \|y\|$ whenever $|x| \leq |y|$, where the absolute value $|x|$ of $x \in X$ is defined by $|x| = x \vee (-x)$.*

Put

$$x^+ = x \vee 0\ ,\ x^- = -(x \wedge 0) \quad \text{and} \quad |x| = x^+ + x^-.$$

It follows that $x = x^+ - x^-$.

Definition 1.4.5 *A subset A of X is p-bounded if there is a positive constant M such that $p(x) \leq M$ for all $x \in A$.*

Note that if a set A is p-bounded and is $\overline{p}$-bounded, then A is p^s-bounded.

Lemma 1.4.1 *Let (X, p) be an asymmetric normed linear space. We have*

$$B_\varepsilon^{p^s}(x) + \theta_0 \subset B_\varepsilon^p(x),$$

for all $\varepsilon > 0$ and $x \in X$.

Proof. In fact, if $y \in B^{p^s}_\varepsilon(x)+\theta_0$, then there is $x_0 \in B^{p^s}_\varepsilon(x)$ and $z_0 \in \theta_0$ such that $y = x_0+z_0$. Note that

$$p(y - x_0) = p(z_0) = 0 \quad \text{and} \quad p(x_0 - x) < \varepsilon.$$

Then we have that

$$\begin{aligned} p(y - x) &\leq p(y - x_0) + p(x_0 - x) \\ &< \varepsilon. \end{aligned}$$

Hence $y \in B^p_\varepsilon(x)$. ■

Definition 1.4.6 *Let (X, p) be an asymmetric normed linear space. We say that it is right-bounded if there exists $r > 0$ such that*

$$rB_p \subset B_{p^s} + \theta_0.$$

Example 1.4.1 *Let $(X, \|.\|, \leq)$ be a Banach lattice and consider the asymmetric normed linear space (X, p), where*

$$p(x) := \|x \vee 0\|, x \in X.$$

We have that (X, p) is right-bounded with constant $r = 1$. We must prove that

$$B_p \subset B_{p^s} + \theta_0.$$

Let $x \in B_p$. Since X is a lattice, there are a positives elements

$$x^+ = x \vee 0 \textit{ and } -x^- = -(x \wedge 0) \textit{ such that } x = x^+ + x^-.$$

Note that

$$p(x^+) = \|x \vee 0\| = p(x) < 1 \quad \textit{and} \qquad p(-x^+) = 0,$$

thus

$$p^s(x^+) = \max\{p(x^+), p(-x^+)\} = p(x^+) < 1,$$

and

$$p(x^-) = \left\|x^- \vee 0\right\| = 0,$$

so $x^+ \in B_{p^s}$ and $x^- \in \theta_0$. This proves the result.

In the case of an asymmetric normed space, there are several completeness notions, which we present following [14], starting with the definitions of Cauchy sequences, of course a sequence $(x_n)_n$ in asymmetric normed space (X, p) is τ_p-convergence to $x \in X$ if

$$\lim_{n \longrightarrow \infty} p(x_n - x) = 0.$$

Definition 1.4.7 *Let (X, p) be an asymmetric normed space. A sequence $(x_n)_n$ in (X, p) is called:*

- Left p-Cauchy if for every $\varepsilon > 0$ there exist $x \in X$ and $n_0 \in N$ such that

$$p(x_n - x) < \varepsilon \qquad \text{for all } m, n \geq n_0.$$

- Right p-Cauchy if for every $\varepsilon > 0$ there exist $x \in X$ and $n_0 \in N$ such that

$$p(x - x_n) < \varepsilon \qquad \text{for all } m, n \geq n_0.$$

- p^s-Cauchy if for every $\varepsilon > 0$ there exist $n_0 \in N$ such that

$$p^s(x_n - x_m) < \varepsilon \quad \text{for all } m, n \geq n_0.$$

Definition 1.4.8 *An asymmetric norm space (X, p) is called **bicomplete** if the associated normed space (X, p^s) is complete, i.e., each p^s-Cauchy sequence is convergent.*
*A bicomplete asymmetric normed space (X, p) is called a **biBanach space**.*

Definition 1.4.9 *A topological space X is a **Hausdorff** if for each pair x, y of distinct points of X, there are disjoint open sets U and V such that x belongs to U and y belongs to V.*

Remark 1.4.3 *It is well known that the norm of a normed space obviously defines a Hausdorff topology. This is not the case when we consider an asymmetric normed linear space (see [9] for more details).*

Chapter 2

Compactness in asymmetric normed spaces

L. M. García-Raffi is the first one who to study about compactness in the asymmetric normed space. Under special conditions he was be able to extand some of it in the finite dimensional in 2005 [7]. Then he complete his research with C. Alegre, I. Ferrando and E.A. Sánchez Pérez. In 2008 [2] they made a new article under the title "Compactness in asymmetric normed spaces" that work was such a strong one that contained a lot of good results.

Basic references about this chapter are [7] [2].

2.1 Compactness and finite dimension

Let us recall the following separation axioms. A topological space (X, τ) is called:

- T_0 if for any pair x, y of distinct points in X, at least one of them has a neighborhood not containing the other;
- T_1 if for any pair x, y of distinct points in X, each of them has a neighborhood not containing the other;
- Hausdorff or T_2 if for any pair x, y of distinct points in X, there exist neighborhoods U of x and V of y such that $U \cap V = \emptyset$.

Definition 2.1.1 *Let (X, p) be an asymmetric normed linear space endowed with the topology τ_p. A set $M \subset X$ is said to be compact if it is compact considered as a subspace of X with the induced topology, that is, (M, p) is compact with respect to the topology $\tau_p|_M$. A set M of X is compact if every sequence in M has a convergent subsequence whose limit is in M.*

Lemma 2.1.1 *Given a set $A \subset X$ of an asymmetric normed linear space (X, p), we have that*

$$\bigcup_{x \in A} \theta_x = A + \theta_0,$$

where

$$A + \theta_0 = \{z \in X : z = x + y, \quad x \in A \text{ and } \ y \in \theta_0\}.$$

Proof. *(1) Let $z \in \bigcup_{x \in A} \theta_x$. Then there exists an $x \in A$ such that $p(z - x) = 0$. This implies that $y \in \theta_0$ with $y = z - x$, and $z \in A + \theta_0$. Then*

$$\bigcup_{x \in A} \theta_x \subset A + \theta_0.$$

(2) Let $z \in A + \theta_0$. Then there exists an $x \in A$ and $y \in \theta_0$ such that $z = x + y$ and then we can express y as $y = z - x$. Then we have

$$p(z - x) = p(y) = 0,$$

so $z \in \theta_x$. Which proves that

$$A + \theta_0 \subset \bigcup_{x \in A} \theta_x,$$

as desired. ■

Lemma 2.1.2 *Let (X, p) be an asymmetric normed linear space and $x \in X$. Then*

$$V_\epsilon(x) = V_\epsilon(x) + \theta_0.$$

Proof. (1) It is obvious that $V_\epsilon(x) \subset V_\epsilon(x) + \theta_0$.

(2) Let $z \in V_\epsilon(x) + \theta_0$. Then there exists $y \in V_\epsilon(x)$ and $w \in \theta_0$ such that $z = y + w$. Then

$$\begin{aligned} p(z - x) &= p(y + w - x) \\ &\leq p(y - x) + p(w) \\ &< \varepsilon. \end{aligned}$$

This implies that $z \in V_\epsilon(x)$. This proves the second inclusion. ■

Lemma 2.1.3 *Let (X, p) be an asymmetric normed linear space and $A \subset X$ an open set. Then*

$$A = A + \theta_0.$$

Proof. It is obvious that $A \subset A + \theta_0$.

Let $z \in A + \theta_0$. Then $z = x + y$, where $x \in A$ and $y \in \theta_0$. Since A is an open set there exists an $\varepsilon > 0$ such that $V_\epsilon(x) \subset A$. On the other hand, we have that

$$V_\epsilon(x) = V_\epsilon(x) + \theta_0.$$

We conclude that $z \in A$. ■

Lemma 2.1.4 *[7, Lemma 5] Given a family $\{A_i,\ i \in I\}$ of sets in (X, p), then*

$$\bigcup_{i \in I} (A_i + \theta_0) = \left(\bigcup_{i \in I} A_i \right) + \theta_0.$$

Proof. Let $x \in \bigcup_{i \in I}(A_i + \theta_0)$ that means that there exists some i satisfying that $x \in A_i + \theta_0$ such that x can be written as $x = x_i + z$ with $x_i \in A_i$ and $z \in \theta_0$. Then

$$x_i \in \bigcup_{i \in I} A_i \quad \text{and } x \in \left(\bigcup_{i \in I} A_i\right) + \theta_0.$$

If $x \in \left(\bigcup_{i \in I} A_i\right) + \theta_0$ there exists an $x_i \in A_i$ and $z \in \theta_0$ such that $x = x_i + z$ and then

$$x \in \bigcup_{i \in I}(A_i + \theta_0).$$

This proves the second inclusion. ■

Lemma 2.1.5 *Let (X, p) be an asymmetric normed linear space. Then*

$$\overline{B}_{ps} + \theta_0 \subset \overline{B}_p.$$

Proof. Let $g \in \overline{B}_{ps} + \theta_0$. Then we can write $g = f_1 + f_2$ such that $f_1 \in \overline{B}_{ps}$ and $f_2 \in \theta_0$ and we have

$$\begin{aligned} p(g) &\leq p(f_1) + p(f_2) \\ &\leq p(f_1) + 0 \\ &\leq p^s(f_1) \\ &\leq 1. \end{aligned}$$

Hence $g \in \overline{B}_p$. ■

Theorem 2.1.1 *[7, Theorem 9] Let (X, p) be a finite dimensional T_1 asymmetric normed linear space, with base $\{e_1, ..., e_n\}$ and let $(x_k)_{k \in \mathbb{N}}$ be a sequence in X. The following statements are equivalent:*

1. $(x_k)_{k \in \mathbb{N}}$ converges to $x = \lambda_1 e_1 + \lambda_2 e_2 + ... + \lambda_n e_n$ with respect to p.

2. The i-co-ordinate sequence of $(x_k)_{k \in \mathbb{N}}$ converges to λ_i, with respect to the Euclidean norm, $i = 1, ..., n$.

Definition 2.1.2 *An asymmetric normed linear space (X, p) is called normable if there is a norm $\|\cdot\|$ on the linear space X such that the topologies τ_p and $\tau_{d_{\|\cdot\|}}$ coincide on X.*

Corollary 2.1.1 *Let (X,p) be a finite dimensional T_1 asymmetric normed linear space. Then (X,p) is normable by the norm p^s.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X that converges to a point x with respect to a p. By Theorem 2.1.1, The i-co-ordinate sequence of $(x_k)_{k\in\mathbb{N}}$ converges to α_k, with respect to the Euclidean norm, $i=1,...,n$. Given a positive real number $M>0$ and an $\frac{\varepsilon}{n}$, there is a k_0^i such that when $k\geq k_0^i$ we have

$$|\alpha_{k,i}-\alpha_k|<\frac{\varepsilon}{nM}.$$

Let $k_0=\max\left\{k_0^i, i=1,...,n\right\}$. Then, if $k\geq k_0$,

$$\begin{aligned}
\overline{p}\left(x_i-x\right) &= \overline{p}\left(\sum_{k=1}^{m}\left(\alpha_{k,i}-\alpha_k\right)e_k\right)\\
&\leq p^s\left(\sum_{k=1}^{m}\left(\alpha_{k,i}-\alpha_k\right)e_k\right)\\
&\leq \sum_{k=1}^{m}p^s\left(\left(\alpha_{k,i}-\alpha_k\right)\left(e_k\right)\right)\\
&\leq \sum_{k=1}^{m}M\left|\alpha_{k,i}-\alpha_k\right|\\
&\leq \varepsilon,
\end{aligned}$$

where we have used the fact that p^s is a norm and equivalent to the Euclidean norm with the constant M. Thus, $(x_k)_{k\in\mathbb{N}}$ converges to x with respect to the norm p^s. ■

In particular observe that, by the Corollary 2.1.1, T_1 separation axiom implies T_2 separation axiom in the finite dimensional case.

Theorem 2.1.2 *[7, Theorem 13] The unit ball of a T_1 asymmetric normed linear space (X,p) is compact if and only if (X,p) is finite dimensional.*

Theorem 2.1.3 *[3, Page 145] Let (X,p) be a finite dimensional asymmetric normed linear space. Then X is T_1 if and only if every compact subset of X is closed.*

Theorem 2.1.4 *Let (X,p) be a finite dimensional asymmetric normed linear space. Then (X,p) is normable if and only if each compact set is closed.*

Proof. Suppose that (X,p) is not normable. Then it is not Hausdorff. So there exist a sequence $(x_n)_{n\in\mathbb{N}}$ in X and two points $x,y\in X$ with $x\neq y$ such that $x_n\longrightarrow x$ and $x_n\longrightarrow y$

with respect to the topology τ_p. Since

$$K = \{x\} \cup \{x_n : n \in \mathbb{N}\},$$

is compact in (X, p) (see Proposition 2.2.5) and $y \in \overline{K} - K$, K cannot be closed.

The converse is well known. ■

Note that, in a finite dimensional normed linear space, every compact set is bounded and hence this theorem provides a version of the Heine-Borel Theorem for asymmetric normed linear spaces.

Proposition 2.1.1 *Let (X, p) be a finite dimensional asymmetric normed linear space such that $\overline{B}_p$ is right-bounded. Then it is compact.*

2.2 Preompactness in asymmetric normed linear spaces

Let (X, p) be an asymmetric normed linear space.

Definition 2.2.1 *We say that a subset A of X is* ***precompact*** *in (X, p) if for all $\varepsilon > 0$ there is a finite set $\{a_1, ..., a_n\}$ in A such that*

$$A \subset \bigcup_{i=1}^{n} B_\varepsilon^p(a_i),$$

or, equivalently,

$$\forall a \in A, \exists i \in \{1, ..., n\}, \qquad p(a - a_i) < \varepsilon.$$

Definition 2.2.2 *We say that a subset A of an asymmetric normed linear space (X, p) is* ***outside precompact*** *in (X, p) if for each $\varepsilon > 0$ we can find a finite set of points $\{x_1, ..., x_n\}$ in X such that*

$$A \subset \bigcup_{i=1}^{n} B_\varepsilon^p(x_i).$$

Remark 2.2.1 *A subset A of X is (outside) precompact if and only if for every $\varepsilon > 0$ there exists a finite subset $\{a_1, ..., a_n\}$ of A (resp. of X) such that*

$$A \subset \bigcup_{i=1}^{n} \overline{B}_\varepsilon^p(a_i).$$

Note that in the case of normed spaces precompactness and outside precompactness are equivalent properties.

Proposition 2.2.1 *Let (X,p) be an asymmetric normed linear space and $(x_n)_{n\geq 1}$ a sequence in X. If $(x_n)_{n\geq 1}$ is p-convergent, then the set $\{x_n : n \geq 1\}$ is outside precompact.*

Proof. If $(x_n)_{n\geq 1}$ is p-convergent to some $x_0 \in X$, then for every $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that $p(x_n - x_0) < \varepsilon$, for all $n \geq k$, implying

$$\{x_n : n \geq 1\} \subset \bigcup_{i=0}^{k} B_\varepsilon^p(x_i),$$

it follows that $\{x_n : n \geq 1\}$ is outside precompact. ■

If a set A is precompact in (X,p), then it is outside precompact in (X,p); but the converse is not true in general. In order to prove this we take the following example.

Example 2.2.1 *Consider the space ℓ_∞ with the asymmetric norm*

$$p(x) = \sup_i x_i^+, \quad x = (x_i)_i \in \ell_\infty.$$

Then $p^s(x) = \sup_i |x_i|$. Consider the sequence

$x_1 = (1, 0, 0, ...)$,

$x_2 = (1, 1, 0, ...)$,

$x_n = (1, 1, ..., \underset{(n)}{1}, 0, ...)$.

Let $x = (1, 1, ...)$. Then

$$\begin{aligned} p(x_n - x) &= p\left((0, 0, ..., \underset{(n)}{0}, -1, -1, ...)\right) \\ &= 0, \end{aligned}$$

that is $x_n \xrightarrow{p} x$. Then by Proposition 2.2.1 the set $A = \{x_n : n \in \mathbb{N}\}$ is outside precompact in (X,p). We now prove by contradiction that $\{x_n : n \in \mathbb{N}\}$ is not precompact in (X,p), for $\varepsilon = \frac{1}{2}$ we can find a finite set of indexes $i_1, i_2, ...i_p$ such that

$$\{x_n : n \in \mathbb{N}\} \subset \bigcup_{j=1}^{p} B_{\frac{1}{2}}^p(x_{i_j}).$$

Then if $n > i_p$, there is a some $i_k \in \{i_1, i_2, ..., i_p\}$ such that $x_n \in B_{\frac{1}{2}}^p(x_{i_k})$, but this is not possible because $p(x_n - x_{i_k}) = 1$, showing that $\{x_n : n \in \mathbb{N}\}$ is not contained in $\bigcup_{j=1}^{p} B_{\frac{1}{2}}^p(x_{i_j})$.

Proposition 2.2.2 *Let (X, p) be an asymmetric normed linear space. The following statements are equivalent:*

i) A subset A of X is precompact in (X, p).

ii) For all $\varepsilon > 0$ we can find a finite set of points $\{x_1, ..., x_n\}$ in X such that $A \subset \bigcup_{i=1}^{n} B_{\varepsilon}^{p}(x_i)$ and $B_{\varepsilon}^{\overline{p}}(x_i) \cap A \neq \emptyset$ for all $i \in \{1, ..., n\}$.

Proof. ($\Longrightarrow$) Suppose that A is precompact in (X, p). Then by definition there is a finite set $\{a_1, ..., a_n\}$ in A such that $A \subset \bigcup_{i=1}^{n} B_{\varepsilon}^{p}(a_i)$ and it is obvious that $B_{\varepsilon}^{\overline{p}}(a_i) \cap A \neq \emptyset$.

($\Longleftarrow$) Fix $\varepsilon > 0$ and choose $\{x_1, ..., x_n\}$ in X such that

$$A \subset \bigcup_{i=1}^{n} B_{\frac{\varepsilon}{2}}^{p}(x_i),$$

and

$$B_{\frac{\varepsilon}{2}}^{\overline{p}}(x_i) \cap A \neq \emptyset,$$

for all $i \in \{1, ..., n\}$. Let $a_i \in B_{\frac{\varepsilon}{2}}^{\overline{p}}(x_i) \cap A$; we must prove $B_{\frac{\varepsilon}{2}}^{p}(x_i) \subset B_{\varepsilon}^{p}(a_i)$. If $x \in B_{\frac{\varepsilon}{2}}^{p}(x_i)$, then

$$\begin{aligned} p(x - a_i) &\leq p(x - x_i) + p(x_i - a_i) \\ &< \tfrac{\varepsilon}{2} + \overline{p}(a_i - x_i) \\ &< \varepsilon. \end{aligned}$$

This implies that, the set A is precompact in (X, p). ■

Let (X, p) be an asymmetric normed linear space. Let a subset A of X. If A is precompact in (X, p^s), then A is precompact in (X, p) and A is precompact in $(X, \overline{p})$ but this condition is not sufficient, as we show in the following example.

Example 2.2.2 *In $X = \ell_\infty$ we define the asymmetric norm*

$$p\left((\alpha_i)_{i=1}^{\infty}\right) := \|(\alpha_i)_{i=1}^{\infty} \vee 0\| = \sup_{i \in \mathbb{N}} \alpha_i^{+} = \sup_{i \in \mathbb{N}} \max\{\alpha_i, 0\}.$$

We have that the unit ball $\overline{B}_{p^s}$ is precompact in (ℓ_∞, p) and in $(\ell_\infty, \overline{p})$ but it is not precompact in (ℓ_∞, p^s). Take $x_0 = (1, 1, ...) \in \overline{B}_{p^s}$ and $x_1 = (-1, -1, ...) \in \overline{B}_{p^s}$ and we show that

$$\overline{B}_{p^s} \subset B_{\varepsilon}^{p}(x_0) \quad \textit{and} \quad \overline{B}_{p^s} \subset B_{\varepsilon}^{\overline{p}}(x_0).$$

Let $\alpha = (\alpha_i)_{i=1}^{\infty} \in \overline{B}_{p^s}$. *We have* $p(\alpha) \leq p^s(\alpha) \leq 1$, *this implies that*

$$p(\alpha_i - 1) < \varepsilon, \text{ for all } i \in \mathbb{N} \text{ and } \varepsilon > 0,$$

and then $p(\alpha - x_0) < \varepsilon$. *On the other hand, the same way we get* $\overline{p}(\alpha_i + 1) < \varepsilon$. *We conclude that* $\overline{B}_{p^s}$ *is precompact in* (ℓ_∞, p) *and precompact in* $(\ell_\infty, \overline{p})$.

Suppose that $\overline{B}_{p^s}$ *is precompact in* (ℓ_∞, p^s), *then* $\overline{B}_{p^s}$ *is compact in the infinite dimensional Banach space* (ℓ_∞, p^s). *This is a contradiction.*

The purpose of the following example is proof that the p-closure of a precompact set in (X,p) is not neecessary precompact in (X,p).

Example 2.2.3 *Consider the asymmetric normed linear space* $(\mathbb{R}, p)$, *where*

$$p(x) = x^+ = x \vee 0, x \in \mathbb{R}.$$

The subset $\mathbb{R}^- := \{x \in \mathbb{R} : x < 0\}$ *is precompact in* (X,p) *because for all* $\varepsilon > 0, \mathbb{R}^- \subset B_\varepsilon^p(-\varepsilon)$, *but* $\overline{\mathbb{R}^-}^p = \mathbb{R}$, *which is not precompact in* (X,p).

The following result shows that the closure with respect to another topology of a precompact set in (X,p) is precompact in (X,p).

Proposition 2.2.3 *Let* (X,p) *be an asymmetric normed linear space. A subset* A *of* X *is (outside) precompact in* (X,p) *if and only if the set* $\overline{A}^{\overline{p}}$ *is (outside) precompact in* (X,p).

Proof. $(\Rightarrow)$ Suppose that A is precompact in (X,p). Let $\varepsilon > 0$. Then, there exists a finite set $\{x_1, ..., x_n\}$ in A such that

$$A \subset \bigcup_{i=1}^{n} B_{\varepsilon/2}^p(x_i) \subset \bigcup_{i=1}^{n} \overline{B}_{\varepsilon/2}^p(x_i).$$

Then

$$\overline{A}^{\overline{p}} \subset \overline{\bigcup_{i=1}^{n} \overline{B}_{\varepsilon/2}^p(x_i)}^{\overline{p}} = \bigcup_{i=1}^{n} \overline{\overline{B}_{\varepsilon/2}^p(x_i)}^{\overline{p}} = \bigcup_{i=1}^{n} \overline{B}_{\varepsilon/2}^p(x_i) \subset \bigcup_{i=1}^{n} B_\varepsilon^p(x_i).$$

$(\Longleftarrow)$ Now, if $\overline{A}^{\overline{p}}$ is precompact in (X,p), then for given $\varepsilon > 0$ there exists a finite subset $\{x_1, ..., x_n\}$ in $\overline{A}^{\overline{p}}$ such that

$$A \subset \overline{A}^{\overline{p}} \subset \bigcup_{i=1}^{n} B_{\varepsilon/2}^p(x_i).$$

Then for a fixed index i there is some $a_i \in A$ such that

$$p(x_i - a_i) = \overline{p}(a_i - x_i) < \frac{\varepsilon}{2}.$$

Now we have to prove that $B^p_{\varepsilon/2}(x_i) \subset B^p_\varepsilon(a_i)$.

Let $y \in B^p_{\varepsilon/2}(x_i)$. Then $p(y - x_i) < \frac{\varepsilon}{2}$ and

$$\begin{aligned} p(y - a_i) &\leq p(y - x_i) + p(x_i - a_i) \\ &< \varepsilon. \end{aligned}$$

So we obtain that A is precompact in (X, p). ■

Corollary 2.2.1 *Let A and B be two subspaces of (X, p) such that $A \subset B$ and B is precompact in (X, p). If A is $\overline{p}$-dense in B then A is precompact in (X, p).*

Proof. If A is $\overline{p}$-dense in B, we have that $\overline{A}^{\overline{p}} = B$. Since B is precompact in (X, p), then $\overline{A}^{\overline{p}}$ is precompact in (X, p), implying A is precompact in (X, p). ■

Proposition 2.2.4 *A compact asymmetric normed linear space (X, p) is separable.*

Proof. Let $A_n \subset X$ be a finite subset such that

$$X = \bigcup_{a \in A_n} B^p_{\frac{1}{n}}(a), \quad n \in N.$$

Obviously, $A = \bigcup_{n=1}^{\infty} A_n$ is countable. For an arbitrary $\varepsilon > 0$ let $n \in \mathbb{N}$ be such that $\frac{1}{n} < \varepsilon$. Then for $x \in X = \bigcup_{a \in A_\varepsilon} B^p_{\frac{1}{n}}(a)$ there exists $a \in A_n \subset A$ such that $x \in B^p_{\frac{1}{n}}(a) \subset B^p_\varepsilon(a)$, proving the p-density of A in X. ■

Proposition 2.2.5 *If a sequence $\{x_n\}_{n=1}^{\infty}$ in (X, p) is p-convergent and x_0 belongs to $\lim_n x_n$, then the set $\{x_0\} \cup \{x_n : n \in \mathbb{N}\}$ is precompact in (X, p).*

Proof. Since $\{x_n\}_{n=1}^{\infty}$ is p-convergent to x_0, then for $\varepsilon > 0$ there is some $n_0 \in \mathbb{N}$ such that $x_n \in B^p_\varepsilon(x_0)$ for all $n \geq n_0$. This implies that

$$\{x_0\} \cup \{x_n : n \in \mathbb{N}\} \subset \bigcup_{i=1}^{n_0} B^p_\varepsilon(x_i).$$

as desired. ■

The inclusion of a limit point is necessary for Proposition 2.2.5, as we saw in Example 2.2.1.

Remark 2.2.2 *It is obvious that if A is precompact in (X,p) then it is p-bounded, but if A is p-bounded does not imply A is precompact in (X,p), to confirm this observation, suppose that the sequence $\{x_n : n \in \mathbb{N}\}$ in Example 2.2.1 is a p-bounded set that is not precompact in (X,p).*

Definition 2.2.3 *The convex hull of a set $A \subset X$ is defined as*

$$conv(A) = \left\{ \sum_{i=1}^{n} \lambda_i x_i, x_i \in A, \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1 \right\}.$$

Remark 2.2.3 *Let (X,p) be an asymmetric normed linear space. The following statements hold:*

1. The finite sum of precompact sets is precompact.

2. The convex hull of a precompact set is precompact.

Proof. Let A_1 and A_2 two precompact sets in (X,p). Let $\varepsilon > 0$. Consider the sets $\{x_1^1, ..., x_n^1\} \subset A_1$ and $\{x_1^1, ..., x_m^1\} \subset A_2$ such that

$$A_1 \subset \bigcup_{i=1}^{n} B^p_{\frac{\varepsilon}{2}}(x_i^1) \text{ and } A_2 \subset \bigcup_{i=1}^{m} B^p_{\frac{\varepsilon}{2}}(x_i^2).$$

Let $z \in A_1 + A_2$, that is $z = z_1 + z_2$ with $z_1 \in A_1$ and $z_2 \in A_2$. There are elements x_i^1 and x_j^2 such that $p(z_1 - x_i^1) < \frac{\varepsilon}{2}$ and $p(z_2 - x_j^2) < \frac{\varepsilon}{2}$. Thus

$$\begin{aligned} p(z - (x_i^1 + x_j^2)) &= p(z_1 - x_i^1 + z_2 - x_j^2)) \\ &\leq p(z_1 - x_i^1) + p(z_2 - x_j^2) \\ &< \varepsilon. \end{aligned}$$

Then $z \in B^p_{\varepsilon}(x_i^1 + x_j^2)$. This means that the set $\{x_i^1 + x_j^2 : i = 1, ..., n, j = 1, ..., m\}$ define an adequate set of centres of p-balls of radius ε to cover the set $A_1 + A_2$. Hence, the finite sum of precompact sets is precompact.

Let A be a precompact subset in (X,p). For $\varepsilon > 0$, we can find a set of points of A, $\{x_1, ..., x_n\}$ such that

$$A \subset \bigcup_{i=1}^{n} B^p_{\frac{\varepsilon}{2}}(x_i) \subset \{x_1, ..., x_n\} + B^p_{\frac{\varepsilon}{2}}(0).$$

Since

$$convex\,(A+B) = convex\,(A) + convex\,(B)\,,$$

we have

$$convex\,(A) \subset convex\,(\{x_1,...,x_n\}) + B^p_{\frac{\varepsilon}{2}}\,(0)\,.$$

On the other hand, since $convex\,(\{x_1,...,x_n\})$ is compact in (X,p^s), then it is precompact in (X,p). Thus we can find a finite set $\{y_1,...,y_n\}$ in $convex\,(\{x_1,...,x_n\})$ such that

$$convex\,(\{x_1,...,x_n\}) \subset \{y_1,...,y_n\} + B^p_{\frac{\varepsilon}{2}}\,(0)\,,$$

and, we can conclude directly that

$$convex\,(A) \subset \{y_1,...,y_n\} + B^p_{\frac{\varepsilon}{2}}\,(0) + B^p_{\frac{\varepsilon}{2}}\,(0) \subset \{y_1,...,y_n\} + B^p_{\varepsilon}\,(0)\,,$$

this confirm that $convex\,(A)$ is precompact in (X,p). ■

Note that if A is precompact in (X,p), then $\overline{convex(A)}^p$ is also precompact in (X,p).

Proposition 2.2.6 *Let (X,p) be an asymmetric normed linear space and $K \subset X$. The subset K is compact respect to the topology generated by p if and only if $K+\theta_0$ is compact for the same topology.*

Proof. Suppose that K is compact. Let be $\{U_i,\ \ i\in I\}$ an open cover of K. We have that

$$K+\theta_0 \subset \bigcup_{i\in I} U_i + \theta_0.$$

By the compactness of K there exists a finite subcover of K, $\{U_j : j\in J\subset I, \quad J \text{ finite}\}$ such that $K \subset \bigcup\limits_{j\in J} U_j$. Then by Lemma 2.1.4 we obtain that

$$K+\theta_0 \subset \bigcup_{j\in J}(U_j+\theta_0).$$

this implies that $K+\theta_0$ admits a finite subcover $\{U_j+\theta_0 : j\in J\subset I, \quad J \text{ finite}\}$. Then $K+\theta_0$ is a compact set.

Now, if $K+\theta_0$ is compact, given an open cover of the set K, $\{U_i,\ \ i\in I\}$, the family $\{U_i+\theta_0,\ \ i\in I\}$ is an open cover of $K+\theta_0$, since $K+\theta_0$ is compact, there exists a finite subcover of $K+\theta_0$, $\{U_j+\theta_0 : j\in J\subset I, \quad J \text{ finite}\}$. Then by lemma 2.1.4,

$$K+\theta_0 \subset \bigcup_{j\in J} U_j + \theta_0.$$

This implies that

$$K \subset \bigcup_{j \in J} U_j,$$

and thus $\{U_j : j \in J \subset I, \quad J \text{ finite}\}$ is a subcover of K obtained from the open cover $\{U_i, i \in I\}$. Hence, K is compact. ■

Corollary 2.2.2 *Given a subset S such that $S \subset K + \theta_0$, if $K + \theta_0$ is cmpact set and $S + \theta_0 = K + \theta_0$ then S is also compact.*

Proof. We have that

$$K + \theta_0 = S + \theta_0.$$

Then $S + \theta_0$ is compact and by previous proposition S is compact. ■

Proposition 2.2.7 *A compact asymmetric normed linear space (X, p) is precompact.*

Proof. The precompactness is obvious: for $\varepsilon > 0$, $\{B^p_\varepsilon(x) : x \in X\}$ is τ_p-open cover of X, so there exists a finite subset A_ε such that

$$X = \bigcup_{a \in A_\varepsilon} B^p_\varepsilon(a),$$

as desired. ■

Remark 2.2.4 *In general, precompactness does not imply compactness.*

And to confirm this observation we get the following example.

Example 2.2.4 *In $\mathbb{R}$ we define the asymmetric norm*

$$p(x) = x^+ = \max\{x, 0\}.$$

The subset

$$\mathbb{R}^- = \{x \in \mathbb{R} : x < 0\},$$

is precompact in (X, p) because for all $\varepsilon > 0$ we have that

$$\mathbb{R}^- \subset B^p_\varepsilon(-\varepsilon),$$

this is because, if $x \in \mathbb{R}^-$ *we have*

$$x < 0 \Longrightarrow x + \varepsilon < \varepsilon \quad \text{and } 0 < \varepsilon,$$

this implies that

$$\begin{aligned} p\left(x - (-\varepsilon)\right) &= (x+\varepsilon)^+ \\ &= \max\{x+\varepsilon, 0\} \\ &< \varepsilon, \end{aligned}$$

thus $x \in B_\varepsilon^p(-\varepsilon)$, *but* $\mathbb{R}^-$ *is not compact in* (X, p) *because there is a* p*-open cover without finite subcover:* $\mathbb{R}^- \subset \bigcup\limits_{n=1}^{\infty} \left]-\infty, -\frac{1}{n}\right[$.

The following result gives a sufficient condition for compactness. This result does not give a characterization of this property (see Example 2.2.5 below). This fact motivates last part in this section.

Proposition 2.2.8 *Let* (X, p) *be an asymmetric normed linear space. If* K *is a subset of* X *such that*

$$K_0 \subset K \subset K_0 + \theta_0,$$

with K_0 *is a compact in* (X, p^s), *then* K *is compact in* (X, p).

Proof. Let $\{A_i\}_{i \in I}$ be a p-open cover of K, we will affirm that there is a finite subcover of K. We have A_i is p^s-open for all $i \in I$. Since K_0 is a subset of K, $\{A_i\}_{i \in I}$ is a p^s-open cover of K_0. Therefore there is $\{A_{i_j}\}_{j=1}^n$ such that

$$K_0 \subset \bigcup_{j=1}^{n} A_{i_j}.$$

We have that

$$\bigcup_{j=1}^{n} \left(A_{i_j} + \theta_0\right) = \left(\bigcup_{j=1}^{n} A_{i_j}\right) + \theta_0 \quad \text{and } A_{i_j} = A_{i_j} + \theta_0, \text{ for all } \; j \in \{1, ..., n\},$$

this implies that,

$$K \subset K_0 + \theta_0 \subset \bigcup_{j=1}^{n} A_{i_j}.$$

This shows that K is compact in (X, p). ∎

The following example proves that there are a compact subsets in (X, p) which do not satisfy the existence of a compact set K_0 in (X, p^s) such that $K_0 \subset K \subset K_0 + \theta_0$.

Example 2.2.5 *In the space (ℓ_1, p) with*

$$p\left((\alpha_i)_{i=1}^{\infty}\right) := \sum_{i=1}^{n} \alpha_i^+.$$

We define the set $K = \{x_n : n \in \mathbb{N}\}$ as

$x_0 = (0, 0, 0, 0, , ...)$,

$x_1 = \left(\frac{1}{2^0}, -1, 0, 0, ...\right)$,

$x_2 = \left(\frac{1}{2^1}, 0, -1, 0, ...\right)$,

$\vdots$

$x_n = \left(\frac{1}{2^{n-1}}, 0, 0, ..., \underset{(n+1)}{-1}, 0, ...\right)$.

We prove first that K is compact in (X, p). If $\{A_i\}_{i \in I}$ is an open cover of K, then there is some index i_0 such that x_0 in A_{i_0} and some radius $\delta > 0$ such that $B_\delta^p(x_0) \subset A_{i_0}$. If $x_n \in K$ then, choosing some n_0 with $\frac{1}{2^{n_0 - 1}} < \delta$, we have that for all $n \geq n_0$, x_n is in $B_\delta^p(x_0) \subset A_{i_0}$ and

$$K \subset \cup_{j=1}^{n_0 - 1} A_{i_j} \cup A_{i_0},$$

where A_{i_j} is an element of the open cover satisfying that $x_j \in A_{i_j}$ for $j = 1, ..., n_0 - 1$.

Now we prove by contradiction that there is not any compact set K_0 in (X, p^s) such that $K_0 \subset K$.

If $K_0 \subset K$ is a compact set in (X, p^s), then either K_0 is finite, or K_0 must contain a p^s-convergent subsequence of the sequence $\{x_n\}_{n=1}^{\infty}$. But the latter is impossible, because the subsequences of $\{x_n\}_{n=1}^{\infty}$ are not p^s-cauchy: if $j \neq k$,

$$\begin{aligned} p^s(x_j - x_k) &= \left|\tfrac{1}{2^j} - \tfrac{1}{2^k}\right| + 1 \\ &> 1. \end{aligned}$$

If K_0 is finite it is not possible that $K \subset K_0 + \theta_0$. Choosing two elements x_n and x_m with $n \neq m$ and $x_m \in x_n + \theta_0$, we must have $p(x_m - x_n) = 0$.

But if $m > n$,

$$p(x_m - x_n) = 1 \neq 0,$$

and for $m < n$,

$$p(x_m - x_n) = \frac{1}{2^m} - \frac{1}{2^n} + 1 \neq 0,$$

that is a contradiction.

Theorem 2.2.1 *[2] Let (X,p) be an asymmetric normed linear space. Let K be a subset of X. Then, if (X,p) is a bi-Banach right-bounded space with constant $r=1$ and K is precompact in (X,p) then there is a compact subset K_0 in (X,p^s) such that*

$$K \subset K_0 + \theta_0.$$

Theorem 2.2.2 *Let (X,p) be an asymmetric normed linear space. Let K be a subset of X. Then, if there is a precompact subset K_0 in (X,p^s) such that $K \subset K_0 + \theta_0$ then K is outside precompact in (X,p).*

Proof. Let $\varepsilon > 0$, we have that K_0 is precompact in (X,p^s), then there is a finite set $\{x_1,...,x_n\}$ in K_0 such that $K_0 \subset \bigcup_{i=1}^{n} B_{\varepsilon}^{p^s}(x_i)$. Then

$$K \subset \bigcup_{i=1}^{n} B_{\varepsilon}^{p^s}(x_i) + \theta_0 = \bigcup_{i=1}^{n} \left(B_{\varepsilon}^{p^s}(x_i) + \theta_0\right) \subset \bigcup_{i=1}^{n} B_{\varepsilon}^{p}(x_i).$$

This implies that K is outside precompact in (X,p). ■

Corollary 2.2.3 *Let (X,p) be a bi-Banach asymmetric normed space and let K be a subset of X. If (X,p) is right-bounded with constant $r=1$ and K is precompact in (X,p) then there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X, p^s-convergent to zero, such that $K \subset \overline{convex\,\{x_n : n \in \mathbb{N}\}}^{p^s} + \theta_0$.*

Proof. By Theorem 2.2.1 there is some compact set K_0 in (X,p^s) such that $K \subset K_0 + \theta_0$. We have that K_0 is compact in (X,p^s), this implies that we can find some p^s-convergent sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $\lim_n p^s(x_n) = 0$ and $K_0 \subset \overline{convex\,\{x_n : n \in \mathbb{N}\}}^{p^s}$ (see Theorem 1.1.2). Then we have

$$K \subset K_0 + \theta_0 \subset \overline{convex\,\{x_n : n \in \mathbb{N}\}}^{p^s} + \theta_0,$$

as desired. ■

Corollary 2.2.4 *Let (X,p) be an asymmetric normed space. Let K be a subset of X. If there is some precompact set K_0 in (X,p^s) such that $K_0 \subset \overline{K}^{\overline{p}}$, and $K \subset K_0 + \theta_0$, then K is precompact in (X,p).*

Proof. Note that we have the conditions to apply the proof of Theorem 2.2.2. Then we can directly conclude that K is outside precompact in (X, p). Thus, for $\varepsilon > 0$ there is a finite set $\{x_1, ..., x_n\}$ in K_0 such that

$$K \subset \bigcup_{i=1}^{n} B_{\varepsilon}^{p^s}(x_i) + \theta.$$

We have that $x_i \in K_0 \subset \overline{K}^{\overline{p}}$, this implies that $B_{\varepsilon}^{\overline{p}}(x_i) \cap K \neq \emptyset$, and by Proposition 2.2.2, K is a precompact set in (X, p). ■

Now, we describe a particular class of subsets of an asymmetric normed linear space (X, p) for which the condition of existence of a compact subset K_0 in (X, p^s) such that

$$K_0 \subset K \subset K_0 + \theta_0, \tag{2.2.1}$$

characterizes the compactness of K in (X, p).
Previously we have proved that, in fact, p-compactness of K does not imply the existence of a subset K_0 satisfying (2.2.1) (see Example 2.2.5). However, it is possible to find a broad class of examples in which these properties are equivalent. Actually, for this to happen it is necessary to impose some strong requirements on the subset K. The technical formulation of this requirement is given by the following definition.

In this part we develop our results under the assumption that (X, p) is right bounded with constant $r = 1$, i.e. $B_{\varepsilon}^{p}(x) = B_{\varepsilon}^{p^s}(x) + \theta_0$ for every $\varepsilon > 0$ and every $x \in X$.

Definition 2.2.4 *Let $K \subset X$ and $C_0 \subset K$. We say that K has the $B(C_0)$-property if there is a function $p : K \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that:*

1. *For every pair $x, y \in C_0$ and for all $t \in \mathbb{R}^+$, $p(x,t) \leq p(y,t)$ whenever $x \in y + \theta_0$.*
2. *$B_{p(x,t)}^{p}(x) \cap K \subset \left(B_t^{p^s}(x) \cap C_0\right) + \theta_0$, for all $t \in \mathbb{R}^+$ and for every $x \in \theta_0$.*

Definition 2.2.5 *Let C_0 and K be two subsets of an asymmetric normed space (X, p), $C_0 \subset K$. We say that K is a C_0-compact set in (X, p) if for all sets of positive real numbers $\{\varepsilon_x\}_{x \in C_0}$, the class $\left\{B_{\varepsilon_x}^{p}(x) : x \in C_0\right\}$ defines a cover of K, and this cover admits a finite subcover.*

Theorem 2.2.3 *Let K be a subset of a bi-Banach asymmetric normed space (X, p). If There is a compact set K_0 in (X, p^s) such that*

$$K_0 \subset K \subset K_0 + \theta_0.$$

Then there is a p^s-closed subset C_0 such that K is C_0-compact in (X, p).

Proof. Take a compact set K_0 in (X, p^s) such that $K_0 \subset K \subset K_0 + \theta$. Then it is p^s-closed, and for every family of radii $\{\varepsilon_x : x \in K_0\}$ the set

$$B^{p^s}_{\varepsilon_x}(x) + \theta_0 = B^{p}_{\varepsilon_x}(x), \quad x \in K_0,$$

define a cover of K because

$$K \subset K_0 + \theta_0 \subset \bigcup_{x \in K_0} B^{p^s}_{\varepsilon_x}(x) + \theta_0 = \bigcup_{x \in K_0} B^{p}_{\varepsilon_x}(x).$$

Since K_0 is compact in (X, p^s) and the set $\left\{B^{p^s}_{\varepsilon_x}(x), x \in K_0\right\}$, give a p^s-cover of K_0, there is a finite p^s-subcover of K_0 defined by a set of points $x_1, x_2, ..., x_n \in K_0$. Since

$$K \subset K_0 + \theta_0 \subset \bigcup_{i=1}^{n} B^{p^s}_{\varepsilon_{x_i}}(x_i) + \theta_0 = \bigcup_{i=1}^{n} B^{p}_{\varepsilon_{x_i}}(x_i),$$

then, K is K_0-compact in (X, p). ∎

Theorem 2.2.4 *[2, Theorem 20] Let K be a subset of a bi-Banach asymmetric normed space (X, p). If there is a p^s-closed subset C_0 such that K is C_0 -compact in (X, p) and has the $B(C_0)$-property with $C_0 \subset K_0$, then there is a compact set K_0 in (X, p^s) such that*

$$K_0 \subset K \subset K_0 + \theta_0.$$

Corollary 2.2.5 *Let K_1 and K be subsets of the asymmetric normed linear space (X, p) with K_1 is compact in (X, p^s) and $K_1 \subset K$. Suppose that there is a p^s-closed subset $C_0 \subseteq K \backslash K_1$ such that $K \backslash K_1$ has the $B(C_0)$-property and is C_0-compact in (X, p). Then there is a compact set K_0 in (X, p^s) such that $K_0 \cup K_1 \subseteq K \subseteq K_0 \cup K_1 + \theta_0$.*

Chapter 3

Continuous and compact operators on asymmetric normed space

In this chapter we study the compact operators on asymmetric normed space. Ş. Cobzaş, from university of "Babes–Bolyai" is the one who did that research in 2006. Also, we mentioned the most important consequences about continuous operators on asymmetric normed space.

Basic references about this chapter are [1, 4, 8].

3.1 Spaces of continuous linear functions

Let (X, p) and (Y, q) be asymmetric normed linear spaces. We will denote by $\mathcal{LC}(X, Y)$ the set of all continuous linear maps from the asymmetric normed linear space (X, p) to the asymmetric normed linear space (Y, q). Also, we will denote by $\mathcal{LC}^s(X, Y)$ the linear space of all continuous linear maps from the normed linear space (X, p^s) to the normed linear space (Y, q^s).

Let us recall that a subset M of a linear space X is **algebraically closed** provided that for

$$x + y \in M, \quad \text{for all } x, y \in M,$$

and

$$\lambda x \in M \qquad \text{for all } x \in M \quad \text{and } \lambda \in \mathbb{R}^+.$$

In this case, we say that M is a semilinear space.

Remark 3.1.1 *The linear space $\mathcal{LC}^s(X, Y)$ together with a norm $(p^s)_q^*$ is a normed linear space, with*

$$(p^s)_q^*(f) = \sup \{q^s(f(x)) : p^s(x) \leq 1\},$$

for all $f \in \mathcal{LC}^s(X, Y)$.

Definition 3.1.1 *A linear map $f : (X, p) \longrightarrow (Y, q)$ is called bounded if there exist positive constant K such that*

$$q(f(x)) \leq K p(x),$$

for all $x \in X$.

Theorem 3.1.1 *A linear map $f : (X, p) \longrightarrow (Y, q)$ is continuous if and only if f is bounded.*

Proof. Suppose that f is bounded. Let $x \in X$ and show that for every $r > 0$,

$$f\left(\overline{B}^p_{\frac{r}{K}}(x)\right) \subset \overline{B}^q_r(f(x)).$$

We have

$$\overline{B}^q_r(f(x)) = \{y \in Y : q(y - f(x)) \leq r\},$$

and

$$f\left(\overline{B}^p_{\frac{r}{K}}(x)\right) = \left\{f(z) : z \in X,\ p(z-x) \leq \frac{r}{K}\right\}.$$

If $y = f(z) \in f\left(\overline{B}^p_{\frac{r}{K}}(x)\right)$,

$$\begin{aligned} q\left(y - f(x)\right) &= q\left(f(z) - f(x)\right) \\ &= q\left(f(z-x)\right) \\ &\leq Kp(z-x) \\ &\leq K\tfrac{r}{K} = r. \end{aligned}$$

Which shows that $y \in \overline{B}^q_r\left(f(x)\right)$.

Conversely, suppose that f is continuous (so continuous on 0), then there exists $r > 0$ such that

$$f\left(\overline{B}^p_r(0)\right) \subset B^q_1(0).$$

So $q\left(f(x)\right) \leq 1$ for all $x \in X$ with $p(x) \leq r$.

If $p(x) \neq 0$ we put

$$z = r\frac{x}{p(x)} \in X.$$

We have $p(z) = r$ and then

$$q\left(f(z)\right) \leq 1. \tag{3.1.1}$$

It means that $q\left(f(x)\right) \leq r^{-1}p\left(x\right)$.

If $p(x) = 0$, then $p(nx) = np(x) = 0$ for all $n \in \mathbb{N}$. From (3.1.1) we can write,

$$q\left(f(x)\right) = \frac{1}{n}q\left(f(nx)\right) \leq \frac{1}{n},\ \text{for all } n \in \mathbb{N}^*,$$

going to the limit when $n \longrightarrow +\infty$, we get $q\left(f(x)\right) = 0$. Which shows that f is bounded with the constant $K = r^{-1}$. ■

Proposition 3.1.1 *If the linear map $f : (X, p) \longrightarrow (Y, q)$ is continuous, then $f : (X, \overline{p}) \longrightarrow (Y, \overline{q})$ is continuous. Hence $\mathcal{LC}(X, Y) \subseteq \mathcal{LC}^s(X, Y)$.*

Proof. Let $f \in \mathcal{LC}(X,Y)$. Then there is $M > 0$ such that $q(f(x)) \leq Mp(x)$ this implies that

$$\begin{aligned}\overline{q}(f(x)) &= q\left(-f(x)\right)\\ &= q\left(f(-x)\right)\\ &\leq Mp(-x)\\ &\leq M\overline{p}(x).\end{aligned}$$

Therefore f is continuous from $(X,\overline{p})$ to $(X,\overline{q})$, hence

$$\begin{aligned}q^s(f(x)) &= \max\left\{q(f(x)),\overline{q}(f(x)\right\}\\ &\leq \max\left\{Mp(x),M\overline{p}(x)\right\}\\ &\leq M\max\left\{p(x),\overline{p}(x)\right\}\\ &\leq Mp^s(x).\end{aligned}$$

We conclude that $\mathcal{LC}(X,Y) \subseteq \mathcal{LC}^s(X,Y)$. ■

Corollary 3.1.1 *Let (X,p) and (Y,q) be two asymmetric normed linear spaces. Then $\mathcal{LC}(X,Y)$ is an algebraically closed subset of $\mathcal{LC}^s(X,Y)$.*

Proposition 3.1.2 *The funtion*

$$p_q^*(f) = \sup\left\{q\left(f(x)\right) : p(x) \leq 1\right\},$$

define an asymmetric norm on the space $\mathcal{LC}(X,Y)$.

Remark 3.1.2 *Note that*

$$p_q^*(f) = \inf\left\{K > 0 : q(f(x)) \leq Kp(x)\right\},$$

and

$$(p^s)_q^*(f) = \inf\left\{K > 0 : q^s(f(x)) \leq Kp^s(x)\right\}.$$

Example 3.1.1 *Let u be an asymmetric norm defined on X by*

$$u(x) = \max\{x,0\}.$$

Let id be the identity function from $(\mathbb{R}, u)$ *into itself. Clearly id is a continuous linear function but* $-id$ *is not continuous, because if* $x < 0, u(-x) = -x$, *so*

$$\begin{aligned} u_u^*(-id) &= \sup\{u(-x) : u(x) \leq 1\} \\ &= \sup\{-x : x \leq 1\} \\ &= \infty. \end{aligned}$$

Thus we conclude that $\mathcal{LC}(X, Y)$ *is not a linear space in general.*

Theorem 3.1.2 *Let* (X, p) *and* (Y, q) *be asymmetric normed linear spaces. Then,*

$$(p^s)_q^*(f) \leq p_q^*(f) \qquad \textit{for all} \quad f \in \mathcal{LC}(X, Y)\ .$$

Proof. Let $f \in \mathcal{LC}(X, Y)$. Then

$$\begin{aligned} q(f(x)) &\leq p_q^*(f)p(x) \\ &\leq p_q^*(f)p^s(x), \end{aligned}$$

and

$$\begin{aligned} q(f(-x)) &\leq p_q^*(f)p(-x) \\ &\leq p_q^*(f)p^s(x), \end{aligned}$$

then

$$\begin{aligned} q^s(f(x)) &= \max\{q(f(x)), q(f(-x))\} \\ &\leq p_q^*(f)p^s(x). \end{aligned}$$

We conclude that

$$(p^s)_q^*(f) \leq p_q^*(f),$$

as desired. ■

Theorem 3.1.3 *[8, Theorem 1] Let* (X, p) *and* (Y, q) *be two asymmetric normed linear spaces. The following statements holds.*

1. $\mathcal{LC}(X, Y)$ is a closed subset of $\left(\mathcal{LC}^s(X, Y), \left(p_q^*\right)^s\right)$.

2. If (Y, q) is a biBanach space, then $\left(\mathcal{LC}^s(X, Y), p_q^*\right)$ is a biBanach space and $\left(\mathcal{LC}(X, Y), p_q^*\right)$ is a biBanach semilinear space.

Theorem 3.1.4 *[1, Corollary 2] Let* $(X, \|.\|)$ *and* $(Y, \|.\|)$ *be two Banach lattices and let* $p(x) = \|x^+\|$ *if* $x \in X$ *and* $p(y) = \|y^+\|$ *if* $y \in Y$, *and let* f *be a linear mapping from* X *to* Y. *Then,* $f \in \mathcal{LC}(X, Y)$ *if and only if* $f \geq 0$.

Next we discuss the preservation by $\left(\mathcal{LC}(X, Y), p_q^*\right)$ of properties as Hausdorffness.

Proposition 3.1.3 *Let* (X, p) *and* (Y, q) *be tow asymmetric normed linear spaces. If* (Y, q) *is Hausdorff, then* $\left(\mathcal{LC}^s(X, Y), p_q^*\right)$ *is Hausdorff.*

Proof. Let $f, g \in \mathcal{LC}^s(X, Y)$ such that $f \neq g$. Then there is $x_0 \in X$ with $f(x_0) \neq g(x_0)$, and we may assume without loss of generality that $p(x_0) \leq 1$. Since (Y, q) is Hausdorff there exists $\varepsilon > 0$ such that

$$B_\varepsilon^q\left(f(x_0)\right) \cap B_\varepsilon^q\left(g(x_0)\right) = \emptyset.$$

We now prove that

$$B_\varepsilon^{p_q^*}(f) \cap B_\varepsilon^{p_q^*}(g) = \emptyset.$$

Let $h \in B_\varepsilon^{p_q^*}(f)$ then $p_q^*(h - f) < \varepsilon$, thus

$$q\left((h - f)(x)\right) < \varepsilon \text{ for all } x \in X \text{ with } p(x) \leq 1,$$

this implies that

$$q\left((h - f)(x_0)\right) < \varepsilon,$$

then $h(x_0) \in B_\varepsilon^q\left(f(x_0)\right)$. We have that $h(x_0) \notin B_\varepsilon^q\left(g(x_0)\right)$ then $q\left((h - g)(x_0)\right) \geq \varepsilon$. Thus $h \notin B_{p_q^*}(g, \varepsilon)$. It follows that

$$B_{p_q^*}(f, \varepsilon) \cap B_{p_q^*}(g, \varepsilon) = \emptyset.$$

We conclude that $\left(\mathcal{LC}^s(X, Y), p_q^*\right)$ is Hausdorff space. ■

Since Hausdorffness is a hereditary property we obtain the following.

Corollary 3.1.2 *Let* (X, p) *and* (Y, q) *be tow asymmetric normed linear spaces. If* (Y, q) *is Hausdorff, then* $\left(\mathcal{LC}(X, Y), p_q^*\right)$ *is Hausdorff.*

3.2 The dual space of an asymmetric normed linear space

Let u be an asymmetric norm defined on X by

$$u(x) = \max\{x, 0\}.$$

Given an asymmetric normed linear space (X, p)

$$X^{s*} = \{f : (X, p^s) \longrightarrow (\mathbb{R}, |.|) : f \text{ is linear and continuous}\},$$

and let

$$X^{*} = \{f : (X, p) \longrightarrow (\mathbb{R}, u) : f \text{ is linear and continuous}\},$$

then we have

$$p^*(f) = p_u^*(f) = \sup\{f(x) \vee 0 : p(x) \leq 1\},$$

for all $\quad f \in X^*$

Clearly X^* is an algebraically closed subset of X^{s*}, and thus it is a semilinear space.

Remark 3.2.1 *If (X, p) is an asymmetric normed linear space, the pair (X^*, p^*) is a biBanach semilinear space. Called the dual space of (X, p).*

3.3 Compact operators on space with asymmetric norm

Let $(X, u), (Y, v)$ be asymmetric normed linear spaces and, let

$$u \in \{p, \overline{p}, p^s\} \quad \text{and} \quad v \in \{q, \overline{q}, q^s\}. \tag{3.3.1}$$

Definition 3.3.1 *A linear operator $T : (X, u) \longrightarrow (Y, v)$ is called compact if the set $T\left(\overline{B}_u\right)$ is precompact in Y.*

We shall denote by $\mathcal{LC}^k(X, Y)$ the set of all linear compact operators from (X, u) to (Y, v).

Proposition 3.3.1 *Let (X, u) and (Y, v) be tow asymmetric normed linear spaces. The following assertions hold.*

1. *$\mathcal{LC}^k(X,Y)$ is a semilinear subspace of $\mathcal{LC}(X,Y)$.*

2. *$\mathcal{LC}^k(X,Y)$ is $(p,\overline{q})$-closed in $\mathcal{LC}(X,Y)$.*

Proof. (1) We give the proof in the case $\mu = p$ and $v = q$.

If $T : (X,p) \longrightarrow (Y,q)$ is compact, then there exists $x_1, ..., x_n \in B_p$ such that

$$\forall x \in B_p, \quad \exists i \in \{1,...,n\}, \quad q\left(T\left(x\right) - T\left(xi\right)\right) \leq 1. \tag{3.3.2}$$

If for $x \in B_p$, $i \in \{1,...,n\}$ is chosen according to (3.3.2), then

$$\begin{aligned} q\left(T\left(x\right)\right) &\leq \quad q\left(T\left(x\right) - T\left(x_i\right)\right) + q\left(T\left(x_i\right)\right) \\ &\leq 1 + \max\left\{q\left(T\left(x_j\right)\right) : 1 \leq j \leq n\right\}, \end{aligned}$$

showing that the operator T is (p,q)-bounded.

Suppose that $T_1, T_2 : (X,p) \longrightarrow (Y,q)$ are compact and let $\varepsilon > 0$. By the compactness of the operator T_1, T_2, there exist $x_1, ..., x_m$ and $y_1, ..., y_n$ in B_p such that

$$\forall x \in B_p, \ \exists i \in \{1,...,m\}, \ q\left(T_1\left(x\right) - T_1\left(x_i\right)\right) \leq \frac{\varepsilon}{2},$$

and

$$\forall x \in B_p, \ \exists j \in \{1,...,n\}, \ \ q\left(T_2\left(x\right) - T_2\left(x_j\right)\right) \leq \frac{\varepsilon}{2}.$$

It follows that for every $x \in B_p$ there exist a pair (i,j) with $1 \leq i \leq m$ and $1 \leq j \leq n$ such that

$$\begin{aligned} q\left(T_1\left(x\right) + T_2\left(x\right) - T_1\left(x_i\right) - T_2\left(x_j\right)\right) &\leq q\left(T_1\left(x\right) - T_1\left(x_i\right)\right) + q\left(T_2\left(x\right) - T_2\left(x_j\right)\right) \\ &\leq \varepsilon. \end{aligned}$$

showing that $\{T_1\left(x_i\right) + T_2\left(x_j\right) : 1 \leq i \leq m, \ 1 \leq j \leq n\}$ is a finite ε-net for $(T_1 + T_2)(B_p)$.

The proof of the compactness of αT, for $\alpha > 0$ and T compact, is immediate.

(2) $\tau\left(p,\overline{q}\right)$-closedness of $\mathcal{LC}^k(X,Y)$.

Let (T_n) be a sequence in $\mathcal{LC}^k(X,Y)$ and let $T \in \mathcal{LC}(X,Y)$ such that (T_n) is $(p,\overline{q})$-convergent to T in $\mathcal{LC}(X,Y)$.

For $\varepsilon > 0$ choose $n_0 \in \mathbb{N}$ such that

$$\forall n \geq n_0, \ \forall x \in B_p, \ \overline{q}\left(T_n\left(x\right) - T\left(x\right)\right) \leq \frac{\varepsilon}{3}. \tag{3.3.3}$$

Let $x_1, ..., x_m \in B_p$ such that $T_{n_0}(x_i), 1 \leq i \leq m$, is an $\varepsilon-$net for $T_{n_0}(B_p)$. Then for every $x \in B_p$ there exists $i \in \{1, ..., m\}$ such that

$$q\left(T_{n_0}(x) - T_{n_0}(x_i)\right) \leq \frac{\varepsilon}{3},$$

so that, by (3.3.3),

$$\begin{aligned} q\left(T(x) - T(x_i)\right) &\leq q\left(T(x) - T_{n_0}(x)\right) + q\left(T_{n_0}(x) - T_{n_0}(x_i)\right) + q\left(T_{n_0}(x_i) - T(x_i)\right) \\ &\leq \varepsilon. \end{aligned}$$

Consequently, $T(x_i)\, 1 \leq i \leq m$, is an $\varepsilon-$net for $T(B_p)$, showing that $T \in \mathcal{LC}^k(X, Y)$. ■

Remark 3.3.1 *The assertion (2) of Proposition 3.3.1 holds for other types of compactness too, i.e.* $\mathcal{LC}^k(X, Y)$ *is* $\tau(u, \overline{v})$*-closed in* $\mathcal{LC}(X, Y)$*.*

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الملخص

يتمثل هذا العمل في دراسة خاصية التراص على الفضاءات تحت تناظرية ناظمية حيث في البداية أدرجنا المفاهيم و النتائج الأساسية لهذا الفضاء ثم تطرقنا لإبراز اهم النتائج المتعلقة بالتراص للفضاءات تحت تناظرية ناظمية. أنهينا العمل بتقديم المؤثرات المتراصة والمؤثرات المستمرة في هذه الفضاءات.

الكلمات المفتاحية: فضاء تحت تناظري ناظمي، مجموعة متراصة، مؤثر متراص، مؤثر مستمر.

Résumé

Ce travail consiste à étudier la compacité sur les espaces asymétriques normés, nous avons inclus des définitions et des résultats de base, ensuite nous avons étudié les résultats les plus importants de la compacité dans les espaces asymétriques normés. Et à la fin de ce travail. Nous avons présenté les opérateurs compacts et les opérateurs continus dans ces espaces.

Mots clés : Espace asymétrique normé, ensemble compact, opérateur compact, opérateur continu.

Abstract

The goal of this memoir is to study the compactness on asymmetric normed linear spaces, we have included basic definitions and results of this space, then, we studied the most important results concerning compactness on the asymmetric normed linear spaces, and the end of this work, we presented the compact and continuous operators in these spaces.

Key words: Asymmetric normed space, compact set, continuous operator, compact operator.